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LIAPUNOV SYSTEMS WITH DAMPING

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A nonlinear, autonomous system of order ($2k+2$) is perturbed by application of damping which is analytic and sufficiently small in norm. The system we consider resembles a Liapunov system [1], in a different sense however to that given in [2]. The perturbed system is transformed in such a manner that the unperturbed system transforms into a quasilinear, nonautonomous system of order $2k$ [3]. If the general solution to the unperturbed system is known, then the process of integration of the system of variational equations can be reduced, according to Poincare [4], to quadratures and this is illustrated with the example of a plane spring pendulum.

1. Transformation of the equations of motion. Consider a class of Liapunov systems (see [1], Sect. 33) with damping, described by the following system of equations:

$$d^2u/d\tau^2 + u - U(u, \dot u, v_1, \ldots, v_k, \dot v_1, \ldots, \dot v_k) = -2\varepsilon F_0(\dot u, \dot v_1, \ldots, \dot v_k) \qquad (1.1)$$

$$d^2v_\varkappa/d\tau^2 + a_{\varkappa 1}v_1 + \ldots + a_{\varkappa k}v_k - V_\varkappa(u, \dot u, v_1, \ldots, v_k, \dot v_1, \ldots, \dot v_k) =$$
$$= -2\varepsilon F_\varkappa(\dot u, \dot v_1, \ldots, \dot v_k) \quad (\varepsilon > 0, \varkappa = 1, \ldots, k)$$

Here a dot denotes a derivative with respect to τ ; $a_{j\varkappa} = a_{\varkappa j}$ $(\varkappa, j = 1,\ldots,k)$ are real constants; U, $V_1,\ldots,V_k$, F_0, $F_1,\ldots,F_k$ are real analytic functions; the expansions for F_0, F_1, $\ldots,F_k$ begin with the terms of at least first order and those for U, $V_1,\ldots,V_k$ with terms of at least second order. We shall assume that the unperturbed system (1.1), i.e. (1.1) in which $\varepsilon = 0$, admits a first integral which must be an analytic function of the variables u, $\dot u$, $v_1,\ldots, v_k$, $\dot v_1,\ldots, \dot v_k$ and have the form [1]

$$H = \dot u^2 + u^2 + W(v_1, \ldots, v_k, \dot v_1, \ldots, \dot v_k) +$$
$$+ S_3(u, \dot u, v_1, \ldots, v_k, \dot v_1, \ldots, \dot v_k) = \mu^2 \quad (\mu > 0) \qquad (1.2)$$

where W is a quadratic form and S_3 is a set of terms of order not lower than the third.

We shall assume that the work done by the selected forces of resistance $F_0, F_1, \ldots, F_k$ over any possible displacement, coinciding in the present case with one of the actual displacements, is negative. In the simplest nonlinear case when $F_j = F(v_j^{\cdot})$ $(j = 0,1, \ldots, k;\ v_0^{\cdot} \equiv u)$ this condition means that $\alpha F(\alpha) > 0$ $(\alpha \neq 0)$. In the linear case it means that the dissipation is complete.

Making the Liapunov substitution

$$u^{\cdot} = \rho\cos\vartheta, \qquad u = \rho\sin\vartheta, \qquad v_\varkappa = \rho z_\varkappa$$
$$v_\varkappa^{\cdot} = \rho z_{k+\varkappa} \qquad (\rho \geqslant 0;\ \varkappa = 1, \ldots, k)$$

under the condition

$$1 - \frac{1}{\rho}\left[U(\rho\sin\vartheta, \rho\cos\vartheta, \rho\mathbf{z}) - 2\varepsilon F_0(\rho\cos\vartheta, \rho\mathbf{z}^{(2)})\right]\sin\vartheta > 0 \tag{1.3}$$

we can reduce the system (1.1) and the first integral of the unperturbed system (1.2) to the form

$$d\rho/d\vartheta = A\,(U\cos\vartheta - 2\varepsilon F_0\cos\vartheta)$$
$$dz_\varkappa/d\vartheta = A\,(z_{k+\varkappa} - \rho^{-1}z_\varkappa U\cos\vartheta + 2\varepsilon\rho^{-1}z_\varkappa F_0\cos\vartheta) \tag{1.4}$$
$$dz_{k+\varkappa}/d\vartheta = A\,(-a_{\varkappa 1}z_1 - \ldots - a_{\varkappa k}z_k - \rho^{-1}z_{k+\varkappa}U\cos\vartheta +$$
$$+\rho^{-1}V_\varkappa + 2\varepsilon\rho^{-1}z_{k+\varkappa}F_0\cos\vartheta - 2\varepsilon\rho^{-1}F_\varkappa) \quad (\varkappa = 1, \ldots, k)$$
$$(A = (1 - \rho^{-1}U\sin\vartheta + 2\varepsilon\rho^{-1}F_0\sin\vartheta)^{-1})$$
$$\rho^2\left[1 + W(\mathbf{z}) + \rho S(\vartheta, \rho, \mathbf{z})\right] = \mu^2 \quad (S = \rho^{-3}S_3) \tag{1.5}$$

Here $\mathbf{z}$ and $\mathbf{z}^{(2)}$ are vectors whose components are $z_1, \ldots, z_k$ and $z_{k+1}, \ldots, z_{2k}$, respectively.

The unperturbed system (1.4), i.e. with $\varepsilon = 0$, can be reduced to a quasilinear, nonautonomous system of the order $2k$ using the integral (1.5). Its solution can be found using the methods of small parameter for sufficiently small values of $\mu > 0$ in (1.5) [3].

2. Complete system of parametric variational equations and its solution. Let us write the system (1.4) in the vector form

$$d\mathbf{x}/d\vartheta = \mathbf{f}(\vartheta, \mathbf{x}; \varepsilon) \tag{2.1}$$

where $\mathbf{x}$ is a vector whose components are $\rho, z_1, \ldots, z_{2k}$; $\mathbf{f}$ is a vector function containing the right-hand sides of the system (1.4) and analytic in $\mathbf{x}$ and ε in the domain of definition of (1.3), and the coefficients of the power series in $\rho, z_1, \ldots, z_{2k}$ are 2π-periodic functions of ϑ. In the following we shall set $2k + 1 = n$, assuming that n is any natural number.

Let us suppose that a solution $\mathbf{x}_0(\vartheta)$ of the unperturbed system (2.1) is known, i.e. when $\varepsilon = 0$

$$d\mathbf{x}_0/d\vartheta = \mathbf{f}(\vartheta, \mathbf{x}_0; 0) \tag{2.2}$$

Using the Poicare theorem [4] we shall seek a solution of (2.1), for sufficiently small $\varepsilon > 0$, in the form

$$\mathbf{x} = \sum_{\nu=0}^{\infty} \varepsilon^\nu \mathbf{x}_\nu(\vartheta) \tag{2.3}$$

Subtracting the identity (2.2) from (2.1) and using the Taylor expansion for a function of two variables, we obtain

$$\sum_{m=1}^{\infty} \varepsilon^m \frac{d\mathbf{x}_m}{d\vartheta} = \sum_{\nu=1}^{\infty} \frac{1}{\nu!}\left(\frac{\partial}{\partial \mathbf{x}} \sum_{m=1}^{\infty} \varepsilon^m \mathbf{x}_m + \frac{\partial}{\partial\varepsilon}\varepsilon\right)^\nu \mathbf{f}(\vartheta, \mathbf{x}_0; 0)$$

Equating the coefficients of like powers of ε, we obtain the following sequence of vector differential equations (a complete system of parametric variational equations)

$$\frac{d\mathbf{x}_1}{d\vartheta} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_0 \mathbf{x}_1 + \left(\frac{\partial \mathbf{f}}{\partial \varepsilon}\right)_0$$

$$\frac{d\mathbf{x}_2}{d\vartheta} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_0 \mathbf{x}_2 + \frac{1}{2}\left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2}\right)_0 \mathbf{x}_1\mathbf{x}_1 + \left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}\,\partial \varepsilon}\right)_0 \mathbf{x}_1 + \frac{1}{2}\left(\frac{\partial^2 \mathbf{f}}{\partial \varepsilon^2}\right)_0$$

$$\frac{d\mathbf{x}_3}{d\vartheta} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_0 \mathbf{x}_3 + \frac{1}{2}\left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2}\right)_0 (\mathbf{x}_1\mathbf{x}_2 + \mathbf{x}_2\mathbf{x}_1) + \left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}\,\partial \varepsilon}\right)_0 \mathbf{x}_2 +$$
$$+ \frac{1}{6}\left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}^3}\right)_0 \mathbf{x}_1\mathbf{x}_1\mathbf{x}_1 + \frac{1}{2}\left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}^2\,\partial \varepsilon}\right)_0 \mathbf{x}_1\mathbf{x}_1 + \frac{1}{2}\left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}\,\partial \varepsilon^2}\right)_0 \mathbf{x}_1 + \frac{1}{6}\left(\frac{\partial^3 \mathbf{f}}{\partial \varepsilon^3}\right)_0$$

. . . .

$$\frac{d\mathbf{x}_m}{d\vartheta} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)_0 \mathbf{x}_m + \frac{1}{2}\left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2}\right)_0 \sum_{\alpha_1+\alpha_2=m} \mathbf{x}_{\alpha_1}\mathbf{x}_{\alpha_2} + \left(\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}\,\partial \varepsilon}\right)_0 \mathbf{x}_{m-1} +$$
$$+ \frac{1}{6}\left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}^3}\right)_0 \sum_{\alpha_1+\alpha_2+\alpha_3=m} \mathbf{x}_{\alpha_1}\mathbf{x}_{\alpha_2}\mathbf{x}_{\alpha_3} + \frac{1}{2}\left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}^2\,\partial \varepsilon}\right)_0 \sum_{\alpha_1+\alpha_2=m-1} \mathbf{x}_{\alpha_1}\mathbf{x}_{\alpha_2} + \qquad (2.4)$$
$$+ \frac{1}{2}\left(\frac{\partial^3 \mathbf{f}}{\partial \mathbf{x}\,\partial \varepsilon^2}\right)_0 \mathbf{x}_{m-2} + \dots + \frac{1}{s!}\left(\frac{\partial^s \mathbf{f}}{\partial \mathbf{x}^s}\right)_0 \sum_{\alpha_1+\dots+\alpha_s=m} \mathbf{x}_{\alpha_1}\dots\mathbf{x}_{\alpha_s} +$$
$$+ \frac{1}{(s-1)!}\left(\frac{\partial^s \mathbf{f}}{\partial \mathbf{x}^{s-1}\partial \varepsilon}\right)_0 \sum_{\alpha_1+\dots+\alpha_{s-1}=m-1} \mathbf{x}_{\alpha_1}\dots\mathbf{x}_{\alpha_{s-1}} + \frac{1}{(s-l)!\,l!}\left(\frac{\partial^s \mathbf{f}}{\partial \mathbf{x}^{s-l}\partial \varepsilon^l}\right)_0 \times$$
$$\times \sum_{\alpha_1+\dots+\alpha_{s-l}=m-l} \mathbf{x}_{\alpha_1}\dots\mathbf{x}_{\alpha_{s-l}} + \dots + \frac{1}{(s-1)!}\left(\frac{\partial^s \mathbf{f}}{\partial \mathbf{x}\,\partial \varepsilon^{s-1}}\right)_0 \mathbf{x}_{m-n+1} + \dots +$$

$$+ \frac{1}{m!}\left(\frac{\partial^m \mathbf{f}}{\partial \mathbf{x}^m}\right)_0 \mathbf{x}_1^m + \frac{1}{(m-1)!}\left(\frac{\partial^m \mathbf{f}}{\partial \mathbf{x}^{m-1}\partial \varepsilon}\right)_0 \mathbf{x}_1^{m-1} + \dots +$$
$$+ \frac{1}{(m-l)!\,l!}\left(\frac{\partial^m \mathbf{f}}{\partial \mathbf{x}^{m-l}\partial \varepsilon^l}\right)_0 \mathbf{x}_1^{m-l} + \dots + \frac{1}{(m-1)!}\left(\frac{\partial^m \mathbf{f}}{\partial \mathbf{x}\,\partial \varepsilon^{m-1}}\right)_0 \mathbf{x}_1 + \frac{1}{m!}\left(\frac{\partial^m \mathbf{f}}{\partial \varepsilon^m}\right)_0$$

Here the subscript $_0$ indicates that the partial derivatives are taken at $\mathbf{x}_0 = \mathbf{x}_0(\vartheta)$ and $\varepsilon = 0$; $\alpha_1, \alpha_2, \dots$ are natural numbers and we have the matrix

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \left\| \frac{\partial f_j}{\partial \xi_h} \right\|_1^n \qquad \left(\mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \right)$$

The consecutive terms in (2.4) must be treated as operators, e.g.

$$\frac{\partial^2 \mathbf{f}}{\partial \mathbf{x}^2} \mathbf{x}_1\mathbf{x}_2 = \left\{ \frac{\partial}{\partial \mathbf{x}} \left[\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right) \mathbf{x}_1 \right] \right\} \mathbf{x}_2$$

If ε appears in (2.1) linearly, i.e. if $\mathbf{f}(\vartheta, \mathbf{x}; \varepsilon) = \mathbf{g}(\vartheta, \mathbf{x}) + \varepsilon \mathbf{h}(\vartheta, \mathbf{x})$, then (2.4) becomes

$$\frac{d\mathbf{x}_1}{d\vartheta} = \left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right)_0 \mathbf{x}_1 + \mathbf{h}(\vartheta, \mathbf{x}_0) \qquad (2.5)$$

$$\frac{d\mathbf{x}_m}{d\vartheta} = \left(\frac{\partial \mathbf{g}}{\partial \mathbf{x}}\right)_0 \mathbf{x}_m + \sum_{\nu=2}^{m} \left[\frac{1}{\nu!} \left(\frac{\partial^\nu \mathbf{g}}{\partial \mathbf{x}^\nu}\right)_0 \sum_{\alpha_1+\dots+\alpha_\nu=m} \mathbf{x}_{\alpha_1}\dots\mathbf{x}_{\alpha_\nu} + \right.$$

$$+\frac{1}{(\nu-1)!}\left(\frac{\partial^{\nu-1}\mathbf{h}}{\partial\mathbf{x}^{\nu-1}}\right)_0 \sum_{\alpha_1+\ldots+\alpha_{\nu-1}=m-1} \mathbf{x}_{\alpha_1}\cdots\mathbf{x}_{\alpha_{\nu-1}}\Bigg] \qquad (m>1)$$

Equations (2.4) and (2.5) are integrated directly term by term only in the scalar case. Poicare has shown [4] however that when the general integral of the unperturbed (i.e. with $\varepsilon = 0$) equation (2.1) is known, then the process of integration of (2.4) and (2.5) which may be of any order, is reduced to quadratures.

Indeed, let $\mathbf{x}_0 = \mathbf{x}_0(\vartheta, \mathbf{a})$ be the general integral of (2.1) with $\varepsilon = 0$, where $\mathbf{a}$ is an n-dimensional vector. Differentiating the identity (2.2) with respect to $\mathbf{a}$ we obtain

$$\frac{d}{d\vartheta}\left(\frac{\partial\mathbf{x}_0}{\partial\mathbf{a}}\right) = \left(\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right)_0 \frac{\partial\mathbf{x}_0}{\partial\mathbf{a}}$$

This implies that $\partial\mathbf{x}_0 / \partial\mathbf{a}$ is the fundamental matrix of each of the homogeneous systems of differential equations corresponding to (2.4) or (2.5). Then we can write the solution of the first of the systems (2.4) or (2.5) with zero initial conditions $\mathbf{x}_1(\vartheta_0) = 0$ in the form

$$\mathbf{x}_1 = \frac{\partial\mathbf{x}_0}{\partial\mathbf{a}} \int_{\vartheta_0}^{\vartheta} \left(\frac{\partial\mathbf{x}_0}{\partial\mathbf{a}}\right)^{-1} \left(\frac{\partial\mathbf{f}}{\partial\varepsilon}\right)_0 d\vartheta \qquad (2.6)$$

Since the solution $\mathbf{x}_0 = \mathbf{x}_0(\vartheta; \mathbf{a})$ is a general one, $\partial\mathbf{x}_0 / \partial\mathbf{a}$ is a nonsingular matrix. For $\mathbf{x}_m$ $(m > 1)$ we obtain formulas analogous to (2.6), in which $(\partial\mathbf{f} / \partial\varepsilon)_0$ appearing under the integral sign is replaced with the inhomogeneous part of the corresponding system (2.4) or (2.5).

3. Example. Let us consider a plane spring pendulum of mass m on a weightless spring of length l in the unstressed state, obeying the Hooke's Law, and of rigidity equal to c (see Fig. 1). Let x and $y' = l + \lambda + y$ be the Cartesian coordinates of the mass m counted from the point O of suspension, $\lambda = mg / c$ denoting the static elongation of the spring. We choose the constant sum of the potential energy Π, the force of gravity and the elastic force of the spring in such a manner that it becomes zero at the position of static equilibrium. We then have

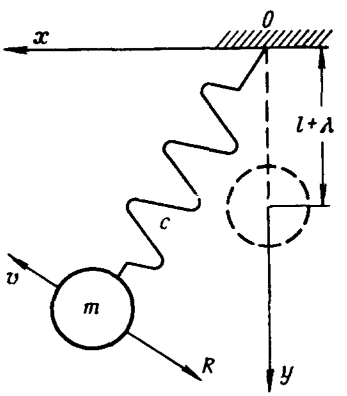

Fig. 1.

$$\Pi = -mgy + \frac{c}{2}\left[\sqrt{x^2 + (l+\lambda+y)^2} - l\right]^2 - \frac{c}{2}\lambda^2, \quad T = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]$$

Let us assume that the mass m is acted upon by an additional reaction force R proportional to velocity. We denote by $\omega = \sqrt{c / m}$ the angular frequency of the vertical oscillations of the mass on the spring and introduce the dimensionless time $\tau = \omega t$ as well as the coordinates $u = y / l$ and $v = x / l$. Then the equations of motion become

$$\frac{d^2u}{d\tau^2} + u - \{(1+\gamma+u)\,[(1+\gamma+u)^2 + v^2]^{-1/2} - 1\} = -2\varepsilon\frac{du}{d\tau} \qquad (3.1)$$

$$\frac{d^2v}{d\tau^2} + \frac{\gamma}{1+\gamma}v - \left\{v\,[(1+\gamma+u)^2 + v^2]^{-1/2} - \frac{v}{1+\gamma}\right\} = -2\varepsilon\frac{dv}{d\tau}$$

where $\gamma = \lambda / l$ and $\varepsilon = {}^1/_2\, b\,(cm)^{-1/2}$ are dimensionless parameters and the expansions of the expressions within the braces begin with the second order terms.

When $\varepsilon = 0$ we have the following energy integral:

$$\frac{1}{cl} \sqrt{2(T + \Pi)c} = \mu \tag{3.2}$$

The Liapunov substitution

$$du/d\tau = \rho \cos\vartheta, \quad u = \rho \sin\vartheta, \quad v = \rho z_1, \quad dv/d\tau = \rho z_2 \tag{3.3}$$

reduces the perturbed system (3.1) and the integral (3.2) of the unperturbed system, to the form (1.4) and (1.5)

$$\begin{gathered}
d\rho/d\vartheta = A(U\cos\vartheta - 2\varepsilon\rho\cos^2\vartheta) \\
dz_1/d\vartheta = A(z_2 - z_1\rho^{-1}U\cos\vartheta + 2\varepsilon z_1\cos^2\vartheta) \\
\frac{dz_2}{d\vartheta} = A[-\gamma(1+\gamma)^{-1}z_1 - z_2\rho^{-1}U\cos\vartheta + \rho^{-1}V - 2\varepsilon z_2\sin^2\vartheta] \\
A = (1 - \rho^{-1}U\sin\vartheta + \varepsilon\sin 2\vartheta)^{-1} \\
U = -\frac{\rho^2 z_1^2}{2(1+\gamma)^2} + O(\rho^3), \quad V = -\frac{\rho^2 z_1 \sin\vartheta}{(1+\gamma)^2} + O(\rho^3)
\end{gathered} \tag{3.4}$$

$$\rho^2\left[1 + \frac{\gamma}{1+\gamma}z_1^2 + z_2^2 + \rho S(\vartheta, \rho, z_1, z_2)\right] = \mu^2 \tag{3.5}$$

where the condition (1.3) is presumed to hold, i.e.

$$1 + \tfrac{1}{2}(1+\gamma)^{-2}\rho z_1^2\sin\vartheta + \varepsilon\sin 2\vartheta + O(\rho^2) > 0 \tag{3.6}$$

The unperturbed system (3.4) (i.e. with $\varepsilon = 0$) admits the generating solution [3]

$$\begin{gathered}
\rho_0(\vartheta;\mu, M, N) = \mu K^{-1/2} + O(\mu^2), \quad z_1^\circ(\vartheta;\mu, M, N) = L(\vartheta) + O(\mu) \\
z_2^\circ(\vartheta;\mu, M, N) = L'(\vartheta) + O(\mu)
\end{gathered} \tag{3.7}$$

$$K = 1 + g^2(M^2 + N^2), \quad g = \sqrt{\frac{\gamma}{1+\gamma}} \quad \left(g \neq \frac{1}{2}\right), \quad L(\vartheta) = M\cos g\vartheta + N\sin g\vartheta$$

$$\mathbf{x}_0 = \begin{pmatrix} \rho_0 \\ z_1^0 \\ z_2^0 \end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix} \mu \\ M \\ N \end{pmatrix}$$

This solution, as was shown in [3] is a general solution for all γ except $\gamma = 1/3$ ($g = 1/2$). Let us compute the elements of (2.6)

$$\frac{\partial \mathbf{x}_0}{\partial \mathbf{a}} = \left\| \begin{matrix} K^{-1/2} + O(\mu) & -g^2 M K^{-3/2}\mu + O(\mu^2) & -g^2 N K^{-3/2}\mu + O(\mu^2) \\ \varphi(\vartheta) + O(\mu) & \cos g\vartheta + O(\mu) & \sin g\vartheta + O(\mu) \\ \psi(\vartheta) + O(\mu) & -g\sin g\vartheta + O(\mu) & g\cos g\vartheta + O(\mu) \end{matrix} \right\|$$

To compute the inverse matrix we shall simplify the result somewhat by setting $\varphi(\vartheta) = \psi(\vartheta) \equiv 0$. Then we have

$$\left(\frac{\partial \mathbf{x}_0}{\partial \mathbf{a}}\right)^{-1} = \left\| \begin{matrix} K^{1/2} + O(\mu) & g^2 K^{-1} L(\vartheta)\mu + O(\mu^2) & K^{-1}L'(\vartheta)\mu + O(\mu^2) \\ O(\mu) & \cos g\vartheta + O(\mu) & -g^{-1}\sin g\vartheta + O(\mu) \\ O(\mu) & \sin g\vartheta + O(\mu) & g^{-1}\cos g\vartheta + O(\mu) \end{matrix} \right\|$$

The vector $(\partial \mathbf{f} / \partial\varepsilon)_0$, where $\mathbf{f}$ is the vector of the right-hand sides of (3.4) and the subscript $_0$ denotes the substitution of $\varepsilon = 0$ and of the generating solution (3.7), is

$$\left(\frac{\partial \mathbf{f}}{\partial \varepsilon}\right)_0 = \begin{pmatrix} -2\mu K^{-1/2}\cos^2\vartheta + O(\mu^2) \\ 2L(\vartheta)\cos^2\vartheta - L'(\vartheta)\sin 2\vartheta + O(\mu) \\ -2L'(\vartheta)\sin^2\vartheta + g^2 L(\vartheta)\sin 2\vartheta + O(\mu) \end{pmatrix}$$

Let us set in (2.6) $\vartheta_0 = 0$. By (3.3) this means that at the initial instant $\tau = 0$ we have $u(0) = 0$. Omitting the obvious calculations performed in accordance with (2.6) we shall write out the first two components of the vector $x_1(\vartheta;\ \mu,\ M,\ N)$ of the first correction

$$\rho_1(\vartheta;\ \mu,\ M,\ N) = -\mu K^{-1/2}\left(\vartheta + \frac{1}{2}\sin 2\vartheta\right) + O(\mu^2)$$

$$z_1^1(\vartheta;\ \mu,\ M,\ N) = \frac{M}{g}\sin g\,\vartheta + \frac{1}{2}M\sin 2\vartheta\cos g\vartheta +$$

$$+ gM\sin^2\vartheta\sin g\vartheta - \frac{1}{2}N\sin 2\vartheta\sin g\vartheta - gN\sin^2\vartheta\cos g\vartheta + O(\mu)$$

It remains to integrate the equation for ϑ [3], and this gives

$$\vartheta = \vartheta(\tau) = \tau + \varepsilon\sin^2\tau + O\ (\mu) + \ldots$$

Here the dots denote the second order terms in μ and ε. Finally we obtain the solution of (3.1) for the case $u(0) = 0$ in the form

$$u = (\rho_0 + \varepsilon\rho_1)\sin\vartheta + O(\varepsilon^2)$$

$$v = \rho_0 z_1^\circ + \varepsilon(\rho_0 z_1^1 + \rho_1 z_1^\circ) + O(\varepsilon^2)$$

The constants μ (initial value of the energy (3.2)) M and N are found from the initial conditions $x(0)$, $x^\cdot(0)$ and $y^\cdot(0)$. The inequality (3.6) defines the bounds for ε and the interval of variation of τ has the order of $O\ (1/\varepsilon)$.

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